Journal of Sound and Vibration (1998) **213**(5), 907–914

COMPLEX NORMAL FORM FOR STRONGLY NON-LINEAR VIBRATION SYSTEMS EXEMPLIFIED BY DUFFING–VAN DER POL EQUATION

A. Y. T. Leung

Manchester School of Engineering, University of Manchester, Manchester M13 9PL, England

AND

Q. C. Zhang

Department of Mechanics, Tianjin University, Tianjin, People's Republic of China

(*Received* 29 *May* 1997, *and in final form* 6 *February* 1998)

We extend the normal form method to study the asymptotic solutions of strongly non-linear oscillators $\ddot{u} + \omega^2 u = f(u, \dot{u})$, where $f(u, \dot{u})$ contains only linear and cubic non-linear terms. The novel contribution is the ansatz $u = \xi + \bar{\xi}$, $\dot{u} = \mathrm{i}\omega_1(\xi - \bar{\xi})$ where ω_1 is to be determined, allowing for the change of the fundamental frequency during the course of vibration, rather than using $u = \xi + \bar{\xi}$, $\dot{u} = \mathrm{i}\omega(\xi - \bar{\xi})$ as suggested by Nayfeh. With the present method, not only the stability of the periodic solutions but also the asymptotic expressions for the periodic solutions can be obtained easily. The results obtained by the method presented coincide very well with the results obtained by numerical integration for the Duffing–van der Pol oscillator with $f(u, \dot{u}) = \mu(1 - u^2)\dot{u} - \beta u^3$. When $\omega = \mu = \beta = 1$, Nayfeh's method gives qualitatively different results from the numerical integration while our method works well even when $\omega = 1$, $\mu = \beta = 3$, since Nayfeh's method is based on weak non-linearities and $\omega = 1$, $\mu = \beta = 3$ is beyond the valid range of assumption.

1. INTRODUCTION

The concept of using co-ordinate transformations to simplify ordinary differential equations has been widely used for a long time. The normal form method is a powerful method to simplify the governing ordinary differential equations. The basic idea of the normal form theory, which was developed to study non-linear vibration problems, is to transform a set of ordinary differential equations into a simpler one by carrying out a formal series transformation; the simplest possible equations are called normal forms. The normal form equations can be solved much more easily than the originals. In recent years, Jezequel [2], Nayfeh [3], and Leung and Zhang [4] have studied weakly non-linear vibration problems by normal form theory. The degenerated and non-degenerated oscillations of autonomous, non-autonomous and parametrically excited systems were studied. Leung and Ge [5] developed a computational method for finding higher order normal forms and gave explicit formulae for Hopf bifurcation analysis. Here the normal form method is extended to study the asymptotic solutions of the strongly non-linear oscillator $\ddot{u} + \omega^2 u = f(u, \dot{u})$, where $f(u, \dot{u})$ contains only linear and cubic non-linear terms. The novel contribution is to express $u = \xi + \bar{\xi}$, $\dot{u} = \mathrm{i}\omega_1(\xi - \bar{\xi})$ where ω_1 is to be determined rather than using $u = \xi + \bar{\xi}$, $\dot{u} = \mathrm{i}\omega(\xi - \bar{\xi})$ as suggested by Nayfeh. With the present method, not

0022–460X/98/250907 + 08 $25.00/0/sv981561

only the stability of the periodic solutions but also the asymptotic expressions for the periodic solutions can be obtained easily. The results obtained by the presented method coincide very well with the results obtained by numerical integration for the Duffing–van der Pol oscillator with $f(u, \dot{u}) = \mu(1 - u^2)\dot{u} - \beta u^3$. When $\omega = \mu = \beta = 1$, Nayfeh's method gives qualitatively different results from the numerical integration while our method works well even when $\omega = 1$, $\mu = \beta = 3\omega = 1$, $\mu = \beta = 3$, since Nayfeh's method is based on weak non-linearities and $\omega = 1$, $\mu = \beta = 3$ is beyond the valid range of assumption.

2. NORMAL FORM FOR STRONGLY NON-LINEAR SYSTEMS

Consider the following second-order non-linear ordinary differential equation,

$$\ddot{u} + \omega^2 u = f(u, \dot{u}), \tag{1}$$

where $f(u, \dot{u})$ contains only linear and cubic non-linear terms. To study the above equation by the complex normal form method, equation (1) is transformed into a differential equation of the first order by complexification in terms of the complex unknown ξ. Let

$$u = \xi + \bar{\xi}, \qquad \dot{u} = \mathrm{i}\omega_1(\xi - \bar{\xi}), \tag{2}$$

where ω_1 is an unknown frequency to be determined. Here ω_1 is chosen as the unknown fundamental frequency rather than ω, as assumed by Nayfeh in reference [3], allowing for the change of the fundamental frequency during the course of vibration. Solving equations (2), one obtains

$$\xi = \frac{1}{2}\left(u - \frac{\mathrm{i}}{\omega_1}\dot{u}\right) \qquad \text{and} \qquad \bar{\xi} = \frac{1}{2}\left(u + \frac{\mathrm{i}}{\omega_1}\dot{u}\right). \tag{3}$$

Differentiating equation (3) with respect to t, gives

$$\dot{\xi} = \frac{1}{2}\left(\dot{u} - \frac{\mathrm{i}}{\omega_1}\ddot{u}\right) = \frac{1}{2}\left(\dot{u} + \frac{\mathrm{i}\omega^2}{\omega_1}u - \frac{\mathrm{i}}{\omega_1}f\right), \tag{4}$$

which, upon using equations (1) and (2), becomes

$$\dot{\xi} = \frac{1}{2}\left(\dot{u} + \frac{\mathrm{i}\omega^2}{\omega_1}u - \frac{\mathrm{i}}{\omega_1}f\right) \tag{5a}$$

$$\dot{\xi} = \mathrm{i}\omega_1\xi + \frac{\mathrm{i}\omega_1}{2}\left(\frac{\omega^2}{\omega_1^2} - 1\right)(\xi + \bar{\xi}) - \frac{\mathrm{i}}{2\omega_1}f. \tag{5b}$$

In order to simplify equation (5), introduce a non-linear transformation from ξ to η in the form [1, 4, 6, 7],

$$\xi = \eta + h(\eta, \bar{\eta}), \tag{6}$$

where η is another complex unknown function of time and h is an odd function to be determined to make the governing equation of η as simple as possible. Substituting equation (6) into equation (5), gives

$$\dot{\eta} = \mathrm{i}\omega_1\eta + \mathrm{i}\omega_1 h - \frac{\partial h}{\partial \eta}\dot{\eta} - \frac{\partial h}{\partial \bar{\eta}}\dot{\bar{\eta}} - \frac{\mathrm{i}}{2\omega_1}f + \frac{\mathrm{i}\omega_1}{2}\left(\frac{\omega^2}{\omega_1^2} - 1\right)(\eta + h + \bar{\eta} + \bar{h}). \tag{7}$$

Because equations (5) involve only linear and third-degree terms, h can be expressed as

$$h = \Delta_1\eta + \Delta_2\bar{\eta} + \Lambda_1\eta^3 + \Lambda_2\eta^2\bar{\eta} + \Lambda_3\bar{\eta}\eta^2 + \Lambda_4\bar{\eta}^3. \tag{8}$$

There are six constants to be determined to fix h. Substituting equation (8) into equation (7), yields a new equation for η. Then the new equation is simplified according to normal form theory. $\dot{\eta}$ has six terms with two resonance terms; u also has six terms with two resonance terms. Eliminating the four non-resonance terms of $\dot{\eta}$ for the simplest possible form and the two resonance terms of u for non-secular solution, gives six equations for the six unknowns in equation (8). Finally, by putting η in polar form, produces one complex equation for the two real unknowns of amplitude a and ω_1 using the steady state condition $\dot{a} = 0$.

In the next section, equations (1) to (8) are applied to a Duffing–van der Pol oscillator.

3. DUFFING–VAN DER POL OSCILLATOR

Consider the Duffing–van der Pol oscillator given by equation (1) with $f(u, \dot{u}) = \mu(1 - u^2)\dot{u} - \beta u^3$, $\mu > 0$ and $\beta > 0$ for the existence of a stable limit cycle. Substituting equation (8) into equation (7), gives

$$\begin{aligned}
\dot{\eta} = {} & \frac{1}{2\omega_1}(\mathrm{i}\omega^2 + \mathrm{i}\Delta_1\omega^2 + \mathrm{i}\bar{\Delta}_2\omega^2 + \mu\omega_1 + \Delta_1\mu\omega_1 - \bar{\Delta}_2\mu\omega_1 + \mathrm{i}\omega_1^2 - \mathrm{i}\Delta_1\omega_1^2 - \mathrm{i}\bar{\Delta}_2\omega_1^2)\eta \\
& + \frac{\mathrm{i}}{2\omega_1}(\omega^2 + \bar{\Delta}_1\omega^2 + \Delta_2\omega^2 + \mathrm{i}\mu\omega_1 + \mathrm{i}\bar{\Delta}_1\mu\omega_1 - \mathrm{i}\Delta_2\mu\omega_1 - \omega_1^2 - \bar{\Delta}_1\omega_1^2 + 3\Delta_2\omega_1^2)\bar{\eta} \\
& + \frac{\mathrm{i}}{2\omega_1}(\beta + \Lambda_1\omega^2 + \bar{\Lambda}_4\omega^2 + \mathrm{i}\mu\omega_1 - \mathrm{i}\Lambda_1\mu\omega_1 + \mathrm{i}\bar{\Lambda}_4\mu\omega_1 - 5\Lambda_1\omega_1^2 - \bar{\Lambda}_4\omega_1^2)\eta^3 \\
& + \frac{\mathrm{i}}{2\omega_1}(3\beta + \Lambda_2\omega^2 + \bar{\Lambda}_3\omega^2 + \mathrm{i}\mu\omega_1 - \mathrm{i}\Lambda_2\mu\omega_1 + \mathrm{i}\bar{\Lambda}_3\mu\omega_1 - \Lambda_2\omega_1^2 - \bar{\Lambda}_3\omega_1^2)\eta^2\bar{\eta} \\
& + \frac{1}{2\omega_1}(3\mathrm{i}\beta + \mathrm{i}\bar{\Lambda}_2\omega^2 + \mathrm{i}\Lambda_3\omega^2 + \mu\omega_1 - \bar{\Lambda}_2\mu\omega_1 + \Lambda_3\mu\omega_1 - \mathrm{i}\bar{\Lambda}_2\omega_1^2 + 3\mathrm{i}\Lambda_3\omega_1^2)\eta\bar{\eta}^2 \\
& + \frac{1}{2\omega_1}(\mathrm{i}\beta + \mathrm{i}\bar{\Lambda}_1\omega^2 + \mathrm{i}\Lambda_4\omega^2 + \mu\omega_1 - \bar{\Lambda}_1\mu\omega_1 + \Lambda_4\mu\omega_1 - \mathrm{i}\bar{\Lambda}_1\omega_1^2 + 7\mathrm{i}\Lambda_4\omega_1^2)\bar{\eta}^3.
\end{aligned} \tag{9}$$

Equation (9) is not exact and is only valid up to third order in η. If $\dot{\eta} \equiv 0$, there are six equations for the six unknown constants of equation (8). However, this is not possible due to the existence of resonance terms. Terms proportional to η, $\eta^2\bar{\eta}$ are resonance terms and hence cannot be eliminated from equation (9), and all the rest should be eliminated [3, 6]. This produces the following four equations.

$$\bar{\eta}: \quad \omega^2 + \bar{\Delta}_1\omega^2 + \Delta_2\omega^2 + \mathrm{i}\mu\omega_1 + \mathrm{i}\bar{\Delta}_1\mu\omega_1 - \mathrm{i}\Delta_2\mu\omega_1 - \omega_1^2 - \bar{\Delta}_1\omega_1^2 + 3\Delta_2\omega_1^2 = 0, \tag{10a}$$

$$\eta^3: \quad \beta + \Lambda_1\omega^2 + \bar{\Lambda}_4\omega^2 + \mathrm{i}\mu\omega_1 - \mathrm{i}\Lambda_1\mu\omega_1 + \mathrm{i}\bar{\Lambda}_4\mu\omega_1 - 5\Lambda_1\omega_1^2 - \bar{\Lambda}_4\omega_1^2 = 0, \tag{10b}$$

$$\eta\bar{\eta}^2: \quad 3\mathrm{i}\beta + \mathrm{i}\bar{\Lambda}_2\omega^2 + \mathrm{i}\Lambda_3\omega^2 + \mu\omega_1 - \bar{\Lambda}_2\mu\omega_1 - \Lambda_3\mu\omega_1 + \mathrm{i}\bar{\Lambda}_2\omega_1^2 + 3\mathrm{i}\Lambda_3\omega_1^2 = 0, \tag{10c}$$

$$\bar{\eta}^3: \quad \mathrm{i}\beta + \mathrm{i}\bar{\Lambda}_1\omega^2 + \mathrm{i}\Lambda_4\omega^2 + \mu\omega_1 - \bar{\Lambda}_1\mu\omega_1 + \Lambda_4\mu\omega_1 - \mathrm{i}\bar{\Lambda}_1\omega_1^2 + 7\mathrm{i}\Lambda_4\omega_1^2 = 0. \tag{10d}$$

Two additional equations are obtained by eliminating the secular terms η, $\eta^2\bar{\eta}$ of u by substituting equations (8) and (6) into the first of equations (2),

$$\begin{aligned}
u = {} & \eta + \bar{\eta} + (\Delta_1 + \bar{\Delta}_2)\eta + (\bar{\Delta}_1 + \Delta_2)\bar{\eta} + (\Lambda_1 + \bar{\Lambda}_4)\eta^3 + (\Lambda_2 + \bar{\Lambda}_3)\eta^2\bar{\eta} \\
& + (\bar{\Lambda}_2 + \Lambda_3)\bar{\eta}\eta^2 + (\bar{\Lambda}_1 + \Lambda_4)\bar{\eta}^3.
\end{aligned} \tag{11}$$

The following conditions should be satisfied in order to eliminate the secular terms η, $\eta^2\bar{\eta}$ of u in equation (11),

$$\Delta_1 = -\bar{\Delta}_2, \qquad \Lambda_2 = -\bar{\Lambda}_3. \tag{12}$$

Taking the conjugate of the left side of equation (10b), multiplying by i and subtracting equation (10d), gives

$$\bar{\Lambda}_1 = -2\Lambda_4. \tag{13}$$

Substituting equations (12) and (13) into equations (10), one gets

$$\begin{aligned} &\Delta_1 = \frac{-(\mathrm{i}\omega^2 + \mu\omega_1 - \mathrm{i}\omega_1^2)}{2(-\mu\omega_1 + 2\mathrm{i}\omega_1^2)}, && \Delta_2 = \frac{-(\mathrm{i}\omega^2 - \mu\omega_1 - \mathrm{i}\omega_1^2)}{2(\mu\omega_1 + 2\mathrm{i}\omega_1^2)}, \\ &\Lambda_1 = \frac{2(-\mathrm{i}\beta + \mu\omega_1)}{(\mathrm{i}\omega^2 + 3\mu\omega_1 - 9\mathrm{i}\omega_1^2)}, && \Lambda_2 = \frac{(-3\mathrm{i}\beta + \mu\omega_1)}{2(\mu\omega_1 - 2\mathrm{i}\omega_1^2)}, \\ &\Lambda_3 = \frac{-(3\mathrm{i}\beta + \mu\omega_1)}{2(\mu\omega_1 + 2\mathrm{i}\omega_1^2)}, && \Lambda_4 = -\frac{\mathrm{i}\beta + \mu\omega_1}{-\mathrm{i}\omega^2 + 3\mu\omega_1 + 9\mathrm{i}\omega_1^2}. \end{aligned} \tag{14}$$

With these choices, equation (9) has the simplest form which contains the resonance terms only and is readily solvable:

$$\dot{\eta} = \frac{\mathrm{i}(\omega^2 + \omega_1^2)}{\mathrm{i}\mu + 2\omega_1}\eta + \frac{3\mathrm{i}\beta - \mu\omega_1}{\mathrm{i}\mu + 2\omega_1}\eta^2\bar{\eta}. \tag{15}$$

Equation (11) has the simplest form up to the third degree,

$$u = \eta + \bar{\eta} + \tfrac{1}{2}\Lambda_1\eta^3 - \Lambda_4\bar{\eta}^3. \tag{16}$$

That is, in view of equation (13),

$$u = \eta + \bar{\eta} - \mathrm{Re}\,(\Lambda_4)(\eta^3 + \bar{\eta}^3) - \mathrm{Im}\,(\Lambda_4)(\eta^3 - \bar{\eta}^3)/\mathrm{i}. \tag{17}$$

Expressing η in polar form

$$\eta = \tfrac{1}{2}a\, \mathrm{e}^{\mathrm{i}\omega_1 t} \tag{18}$$

gives an equation for ω_1 using the steady state condition $\dot{a} = 0$.

Substituting the polar form (18) into equation (15), produces

$$\mathrm{i}\omega_1\eta + \frac{\dot{a}}{a}\eta = \frac{\mathrm{i}(\omega^2 + \omega_1^2)}{\mathrm{i}\mu + 2\omega_1}\eta + \frac{3\mathrm{i}\beta - \mu\omega_1}{\mathrm{i}\mu + 2\omega_1}\eta\,\frac{a^2}{4}. \tag{19}$$

Multiplying both sides of equation (19) with $\mathrm{i}\mu + 2\omega_1$ and eliminating η, gives

$$\left(\mathrm{i}\omega_1 + \frac{\dot{a}}{a}\right)(\mathrm{i}\mu + 2\omega_1) = \mathrm{i}(\omega^2 + \omega_1^2) + (3\mathrm{i}\beta - \mu\omega_1)\,\frac{a^2}{4}. \tag{20}$$

Separating the real and imaginary parts in the above equation gives

$$-\omega_1\mu + 2\,\frac{\dot{a}}{a}\,\omega_1 = -\mu\omega_1\,\frac{a^2}{4}, \tag{21}$$

$$2\omega_1^2 + \frac{\dot{a}}{a}\,\mu = (\omega^2 + \omega_1^2) + 3\beta\,\frac{a^2}{4}. \tag{22}$$

Solving equation (21), gives

$$\dot{a} = \frac{\mu}{2} a(1 - \tfrac{1}{4}a^2). \tag{23}$$

To obtain the steady-state periodic solution, let $\dot{a} = 0$ in equation (23). Then

$$a = 2. \tag{24}$$

Substituting $a = 2$ into equation (22), gives

$$\omega_1^2 = \omega^2 + 3\beta. \tag{25}$$

The stability of the solution is now checked. Since $\dot{a}$ is greater than zero for $a < 2$, and $\dot{a}$ is less than zero for $a > 0$ in equation (23), then, $a > 2$ is a stable solution.

Substituting equation (18) into equation (17) with equations (24) and (25), one gets

$$\begin{cases} u = 2\cos(\omega_1 t) + A[B_1 \cos(3\omega_1 t) + B_2 \sin(3\omega_1 t)] \\ \omega_1^2 = \omega^2 + 3\beta \end{cases}, \tag{26}$$

where

$$A = 2/(\omega^4 + 9\mu^2\omega_1^2 - 18\omega^2\omega_1^2 + 81\omega_1^4),$$

$$B_1 = -\beta\omega^2 + 9\beta\omega_1^2 + 3\mu^2\omega_1^2,$$

$$B_2 = 3\beta\mu\omega_1 + 1\mu\omega^2\omega_1 - 9\mu\omega_1^3.$$

The steady-state solution of the Duffing–van der Pol oscillator according to the method proposed by Nayfeh [3] is,

$$\begin{cases} u = 2\cos(\omega_1^* t) + \dfrac{\beta}{4\omega^2}\cos(3\omega_1^* t) - \dfrac{\mu}{4\omega}\sin(3\omega_1^* t), \\ \omega_1^* = \omega + \dfrac{3\beta}{2\omega}. \end{cases} \tag{27}$$

Equations (27) were obtained as follows. Comparing equation (1) with equation (1.53) in Nayfeh's book [3] (p. 20), gives

$$\varepsilon\mu = \mu, \qquad \varepsilon\alpha_1 = -\beta, \qquad \varepsilon\alpha_2 = -\mu, \qquad \alpha_3 = \alpha_4 = 0.$$

These values were substituted into equations (1.63), (1.64) and (1.65) from Nayfeh's book. Note that it should be ω rather than ω_0 in equation (1.65). Let $a = 0$ in equation (1.64) and substitute $a = 2$ into equations (1.63) and (1.65), produces equations (27).

If $f = -2n\, \mathrm{d}u/\mathrm{d}t$ then equations (21) and (22) are respectively,

$$\omega_1 2n + 2\frac{\dot{a}}{a}\omega_1 = 0 \qquad \text{and} \qquad 2\omega_1^2 - \frac{\dot{a}}{a}2n = \omega^2 + \omega_1^2.$$

According to equation (21), one gets $a = A\,\mathrm{e}^{-nt}$. According to equations (22) and (21),

$$\omega_1^2 = \omega^2 - 2n^2,$$

one gets the exact solution

$$u = A\,\mathrm{e}^{-nt}\sin(\omega_1 t + \alpha),$$

where $\omega_1 = \sqrt{\omega^2 - n^2}$.

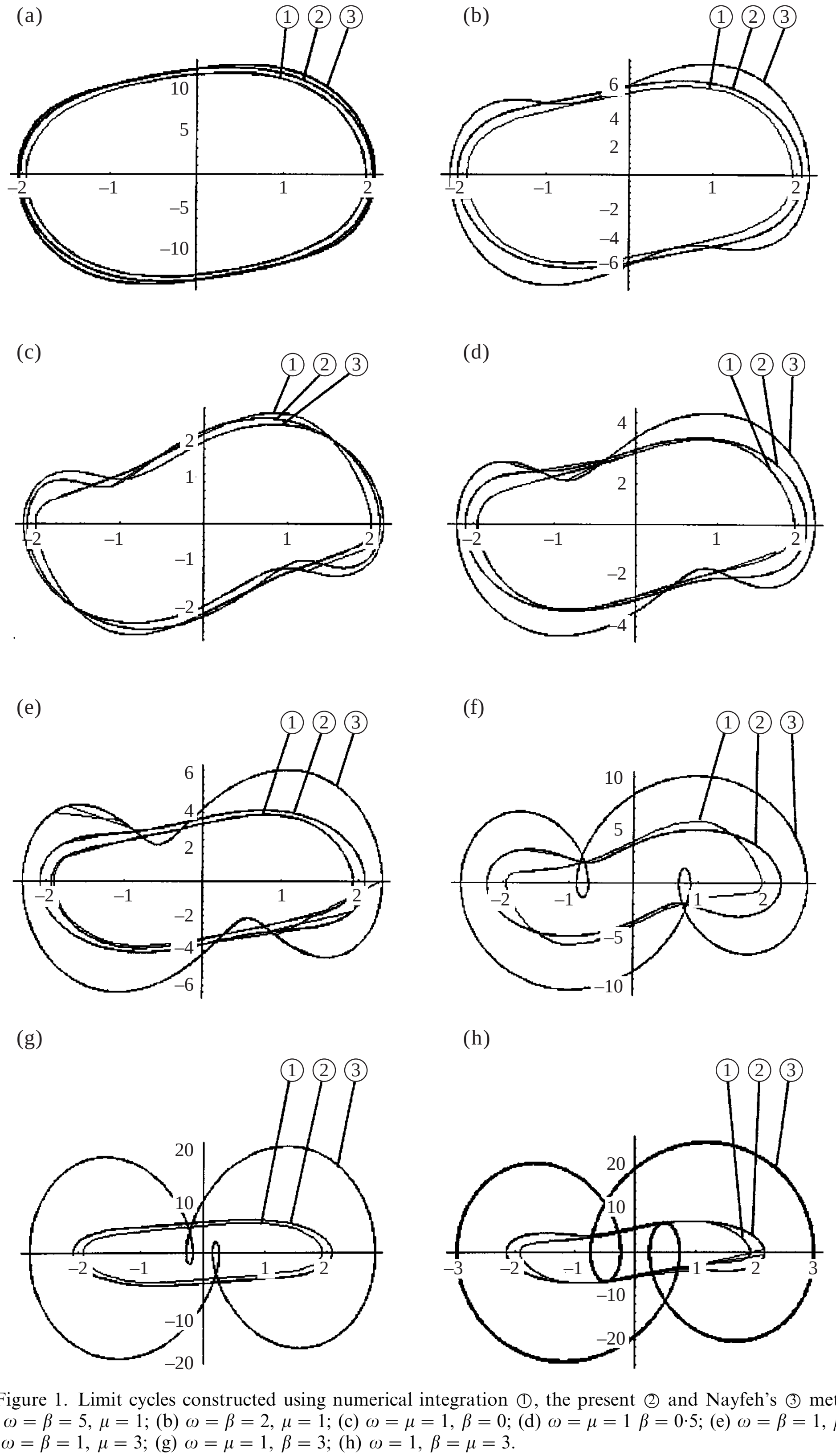

Figure 1. Limit cycles constructed using numerical integration ①, the present ② and Nayfeh's ③ methods. (a) $\omega = \beta = 5$, $\mu = 1$; (b) $\omega = \beta = 2$, $\mu = 1$; (c) $\omega = \mu = 1$, $\beta = 0$; (d) $\omega = \mu = 1$ $\beta = 0{\cdot}5$; (e) $\omega = \beta = 1$, $\mu = 1$; (f) $\omega = \beta = 1$, $\mu = 3$; (g) $\omega = \mu = 1$, $\beta = 3$; (h) $\omega = 1$, $\beta = \mu = 3$.

TABLE 1

Comparison of the resulting periods (s)

Case	Numerical	Present	Nayfeh's
a	1·006	0·993	0·967
b	2·060	1·987	1·795
c	6·690	6·283	6·283
d	4·170	3·974	3·590
e	3·320	3·142	2·513
f	3·690	3·142	2·513
g	2·120	1·987	1·142
h	2·220	1·987	1·142

The phase portraits of a limit circle are shown in Figure 1 for different parameters. Line 1 shows the results obtained by numerical integration, line 2 shows the results obtained by the present method, and line 3 shows the results obtained using Nayfeh's [3] method. In carrying out numerical integration for the steady state solution, the initial conditions $u = 2$ and $\mathrm{d}u/\mathrm{d}t = 0$ were chosen and all solution points before reaching a steady state were disregarded. The diagrams in Figure 1 are arranged in the order of increasing values of the non-linear parameters. For weakly non-linear vibration shown in Figures 1(a)–(d), for the cases (a) $\omega = \beta = 5$, $\mu = 1$, (b) $\omega = \beta = 2$, $\mu = 1$, (c) $\omega = \mu = 1$, $\beta = 0$, and (d) $\omega = \mu = 1$, $\beta = 0{\cdot}5$, the results obtained by the present and Nayfeh's [3] methods coincide well with the results obtained using numerical integration. For increasingly strong non-linear vibration shown in Figures 1(c)–(h) for the cases (e) $\omega = \beta = 1$, $\mu = 1$, (f) $\omega = \beta = 1$, $\mu = 3$, (g) $\omega = \mu = 1$, $\beta = 3$, and (h) $\omega = 1$, $\mu = \beta = 3$, the results obtained by Nayfeh's method [3] are quite different from the results obtained by numerical integration, even the topological structures are different. The respective periods are tabulated in Table 1. Therefore, the method proposed by Nayfeh is not suitable for studying strongly non-linear vibration problems because it is based on the assumption of weak non-linearities. The results obtained by our method coincide quite well with the results obtained by numerical integration even when the non-linear terms are quite strong.

4. CONCLUSION

An efficient and simple method for studying autonomous strongly non-linear vibration systems is proposed. Only simple algebraic computations are needed to calculate the steady state periodic solution for a given system. With the present method, not only the stability of the periodic solutions but also the asymptotic expressions for the periodic solutions can be easily obtained. The results obtained by the presented method coincide quite well with the results obtained using numerical integration even when the non-linear terms are quite strong. The presented method is to be developed further in order to study strongly non-linear non-autonomous vibration systems and strongly non-linear multiple-degree-of-freedom vibration systems.

REFERENCES

1. A. H. NAYFEH and D. T. MOOK 1979 *Nonlinear Oscillations*. New York: John Wiley & Sons.
2. L. JEZEQUEL and C. H. LAMARQUE 1991 *Journal of Sound and Vibration* **149**, 429–459. Analysis of nonlinear dynamical systems by the normal form theory.
3. A. H. NAYFEH 1993 *Method of Normal Forms*. New York: John Wiley & Sons.

4. A. Y. T. LEUNG and ZHANG QICHANG 1994 *Shock and Vibration* **1**, 233–239. Normal form analysis of Hopf bifurcation exemplified by Duffing's equation.
5. A. Y. T. LEUNG and T. GE 1995 *Shock and Vibration* **2**, 307–319. An algorithm for higher order Hopf normal form.
6. J. GUCKENHEIMER and P. HOLMES 1983 *Nonlinear Oscillations*, *Dynamical Systems*, *and Bifurcations of Vector Fields*. New York: Springer-Verlag.
7. D. WANG 1990 *Advances in Mathematics* **19**, 38–71. An introduction to the normal form theory of ordinary differential equations.